THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Tutorial 9
21st November 2024

- Tutorial exercise would be uploaded to blackboard on Mondays provided that there is a tutorial class on that Thursday. You are not required to hand in the solution, but you are advised to try the problems before tutorial classes.
- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

Most rings that we will encounter or use are rings with unity, so unless specified otherwise, "a ring" will mean "a ring with unity".

1. Determine whether the followings are maximal ideals. If not, write down a maximal ideal containing it.
   (a) $\langle -4\rangle \subset \mathbb{Z}$.
   (b) $\langle 7\rangle \subset \mathbb{Z}[x]$.
   (c) $\langle x\rangle \subset \mathbb{Q}[x]$.
   (d) $\langle x^3+1\rangle \subset \mathbb{Q}[x]$.
   (e) $\langle x^2+1\rangle \subset \mathbb{Z}[x]$.
   (f) $\langle x+1, x^2+2x+4\rangle \subset \mathbb{Z}_6[x]$.
   (g) $\langle (3,5)\rangle \subset \mathbb{R}\times\mathbb{R}$.
2. A division ring is a (not necessarily commutative) ring in which every nonzero element has a multiplicative inverse. Prove that if R only has two right ideals, then R is a division ring.
3. Let R be a commutative ring and I is an ideal, we define the radical of an ideal as $\sqrt{I} := \{x \in R : x^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}$, we call an ideal a radical ideal if $\sqrt{I} = I$.
   (a) Verify that $\sqrt{I}$ is indeed an ideal containing I.
   (b) Show that $\sqrt{\sqrt{I}} = \sqrt{I}$.
   (c) Show that any prime ideal is radical.
4. Suppose I, J are ideals so that $I \cap J = \{0\}$, show that for any $a \in I, b \in J$, we have $ab = 0$.
5. Let $f : R \to S$ be a surjective ring homomorphism, suppose $I \subset S$ is a maximal ideal, prove that $f^{-1}(I)$ is again a maximal ideal.
6. Suppose that R is a ring in which there is an $n \in \mathbb{Z}_{>0}$ so that $x^n = x$ for all x. Prove that every prime ideal is maximal.

7. (a) Let R be a commutative ring, let N be the set of nilpotent elements, i.e. $a \in N$ if and only if $a^k = 0$ for some $k \geq 0$. Prove that N is an ideal. (It is called the nilradical)

   (b) Show that N is contained in $\bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$ the intersection of all prime ideals in R.

   (c) Let $a \in R$ be an element that is not nilpotent, prove that there exists some prime ideal $I \subset R$ such that $a \notin I$. Hence conclude that $N = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$. (Hint: consider Q14 of tutorial 8.)

8. Let R be a commutative ring, let $J(R) = \bigcap_{\mathfrak{m}:\text{maximal}} \mathfrak{m}$ to be the intersection of maximal ideals. Prove that $J(R) = \{a \in R |\, 1 - ax \text{ is a unit of } R \text{ for all } x \in R\}$.